On optimal piercing of a square

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Abstract

We treat the following problem: given an $n \times n$ square $ABCD$, determine the minimum number of points that need to be chosen inside the square $ABCD$ such that there does not exist a unit square inside the square $ABCD$ containing none of the chosen points in its interior. In other words, we are interested to know how to most efficiently "destroy" a square-shaped object of side length n, where "destroying" is achieved by piercing as few as possible small holes, and the square is considered "destroyed" if no unpierced square piece of unit side length can be salvaged. This problem actually belongs to the family of problems centered about the so-called *piercing number*: indeed, if $\mathscr{U}_n$ denotes the collection of all open unit squares that can be fitted inside a given $n \times n$ square, the value that we are looking for is the piercing number of the collection $\mathscr{U}_n$, denoted by $\pi(\mathscr{U}_n)$. We show that $\pi(\mathscr{U}_n) = n^2$ when $n \leqslant 7$, and give an upper bound for $\pi(\mathscr{U}_n)$ that is asymptotically equal to $\frac{2}{\sqrt{3}}n^2$, which we believe is asymptotically tight. We then generalize our reasoning in order to obtain a similar upper bound when $ABCD$ is a rectangle, as well as an upper bound for $\pi(\mathscr{U}_x)$ when x is not necessarily an integer. Finally, we show that our results have an application to the problem of packing a given number of unit squares in the smallest possible square; it turns out that our results present a general "framework" based on which we are able to reprove many results on the mentioned problem (originally obtained independently of each other) and also obtain a new result on packing 61 unit squares.

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1 Introduction

There are several lines of research concerning arrangements of unit squares with respect to a larger square, such as packing n unit squares in the smallest possible square [11], or covering the largest possible square with n unit squares [12]. There are also several lines of research concerning arrangements of points inside a given square, such as the problem initiated by Moser [18] to find how large the minimum distance determined by n points in a unit square can be (which is today often researched in its equivalent form of packing circles in a square [3, Section D1] [23]) or the problem of determining the area of the largest convex region not containing in its interior any of n points chosen in a unit square [19, 21].

We hereby study a problem that presents a kind of interplay between these two classes of problems. In fact, it belongs to a (quite general) family of problems centered about the so-called *piercing number*. Namely, given a collection of figures $\mathscr{F}$ in the Euclidean plane (or, more generally, space), the *piercing number* of $\mathscr{F}$, denoted by $\pi(\mathscr{F})$, is defined as the minimum number of points that need to be chosen in such a way that each figure from $\mathscr{F}$ contains at least one of the chosen points (in other words, how many "needles" are required to pierce all members of $\mathscr{F}$). One of the first questions of this kind was asked by Gallai [10, Section III.13]: determine the smallest integer k such that, given any family of circular disks in the plane where every two of them have a common point, there exists a set of k points such that each disk contains at least one of those points; in other words, the value that is asked for equals $\sup_{\mathscr{F}} \pi(\mathscr{F})$, where $\mathscr{F}$ ranges over all the described families of disks. (It is now known that the answer is $k = 4$, where the lower bound is due to Grünabum [14], while a proof of the upper bound had been announced by Danzer in 1954, though the first published proof is due to Stachó [22]; Danzer himself published [5] a proof in 1986, though this is not his original proof.) Various other problems of this kind have been investigated: when the space is d-dimensional, when all the disks are congruent, when the family consists of translates/homothethic images of a given (usually convex) figure etc. (We mention, for example, the result of Karasev [16], who proved that 3 points are always sufficient, and sometimes necessary, to pierce any family of translates of a compact convex set in the plane, any two of which have nonempty intersection.) These problem are usually called Gallai-type problems. A further family of problems that has attracted quite a lot of attention is the family of the so-called (p, q)-problems.

They ask for piercing numbers of finite families of sets in the d-dimensional space, such that among every p members of the family there exist q of them with a nonempty intersection. One of the most important results for this class of problems is proved by Alon and Kleitman [1], who showed that, whenever p, q and d are fixed and $p \geqslant q \geqslant d+1$, then $\pi(\mathscr{F})$ has an upper bound depending only on (p,q,d). However, exact values of $\sup_{\mathscr{F}} \pi(\mathscr{F})$ (for fixed p,q,d) are known only in some very special cases. Apart from $d=1$, when it is known that $p+q-1$ are always sufficient and sometimes necessary to pierce $\mathscr{F}$ (proved by Hadwiger and Debrunner [15], in the paper where this family of problems has actually been introduced), even for $(p,q,d)=(4,3,2)$ it is only known that the supremum is bounded below by 3 (see [4]) and above by 13 (see [17]). For more information about problems related to the piercing number, see the surveys [6, 8, 7].

We treat the following problem: given an $n \times n$ square $ABCD$, determine the minimum number of points that need to be chosen inside the square $ABCD$ such that there does not exist a unit square inside the square $ABCD$ containing none of the chosen points in its interior. In other words, if $\mathscr{U}_n$ denotes the collection of all open unit squares that can be fitted inside a given $n \times n$ square, we are looking for the value $\pi(\mathscr{U}_n)$. The problem can also be presented in the following way: we are interested to know how to most efficiently "destroy" a square-shaped object of side length n, where "destroying" is achieved by piercing as few as possible small holes, and the square is considered "destroyed" if no unpierced square piece of unit side length can be salvaged. Stated like this, it seems that this problem is quite applicable in real life. Furthermore, as it will turn out, it also has a direct application to the already mentioned research problem of packing n unit squares in the smallest possible square.

The work is divided into sections as follows. In Section 2 we show that for $n \in \mathbb{N}$, $n \leqslant 4$, we have $\pi(\mathscr{U}_n) = n^2$ (note that n^2 is a trivial lower bound for $\pi(\mathscr{U}_n)$, and thus we only need to prove that $\pi(\mathscr{U}_n) \leqslant n^2$). In Section 3 we prove an upper bound for $\pi(\mathscr{U}_n)$ asymptotically equal to $\frac{2}{\sqrt{3}}n^2$. Our upper bound actually matches the lower bound n^2 for $n \leqslant 7$, and thus we get a corollary that for $n \leqslant 7$ we have $\pi(\mathscr{U}_n) = n^2$. (This in fact includes the results from Section 2 as a special case. However, in Section 3 we actually use some parts of the proof from Section 2, while the construction given in Section 2 is much more natural and thus we believe that the underlying idea is simpler to understand if seen on that construction first.) In Section

4 we show how the upper bound from Section 3 can be easily generalized to the case when $ABCD$ is a rectangle; we then modify the upper bound from Section 3 in order to obtain an upper bound for $\pi(\mathscr{U}_x)$ when x is not necessarily an integer. In Section 5 we show that our results enable us to reproduce, as a direct consequence, some known results on the square packing problem (among which is a result that the smallest square in which 46 unit squares can be packed is the square of side length 7, which has been proved only recently [2]), and further obtain a new result on packing 61 unit squares. Finally, in the last section we state a conjecture about asymptotical tightness of our upper bound for $\pi(\mathscr{U}_n)$.

Our techniques remind of some ideas often used in the context of "unavoidable points," a notion developed by Friedman [11] in relation to the square packing problem; in fact, some of our proofs can be a little bit shortened by appealing to some lemmas from there. We instead choose to write the paper in a completely self-contained way.

2 The case $n \leqslant 4$

The construction that proves the case $n \leqslant 4$ is actually quite natural, although the proof becomes somewhat technical at some points.

Theorem 1. *For $n \leqslant 4$, $\pi(\mathscr{U}_n) = n^2$.*

Proof. Since the $n \times n$ square can be divided into n^2 interior-disjoint unit squares, it is clear that $\pi(\mathscr{U}_n) \geqslant n^2$. Let us show that n^2 points suffice. We first show this for $n = 4$.

Let the vertices A, B, C, D of the square $ABCD$ have the coordinates $(0,0)$, $(4,0)$, $(4,4)$ and $(0,4)$, respectively. We choose 16 points at the following coordinates:

$$\left(1-\varepsilon+i\frac{2+2\varepsilon}{3}, 1-\varepsilon+j\frac{2+2\varepsilon}{3}\right),\ 0 \leqslant i,j \leqslant 3,$$

where ε is going to be chosen later. That way, the chosen 16 points represent a square lattice with the step $\frac{2+2\varepsilon}{3}$. Let $PQRS$ be the square that bounds this lattice (Figure 1).

We need to show that each unit square inside the square $ABCD$ contains at least one of the chosen points in its interior (for a suitable ε). Let us first consider a unit square whose center is inside the square $PQRS$. Notice

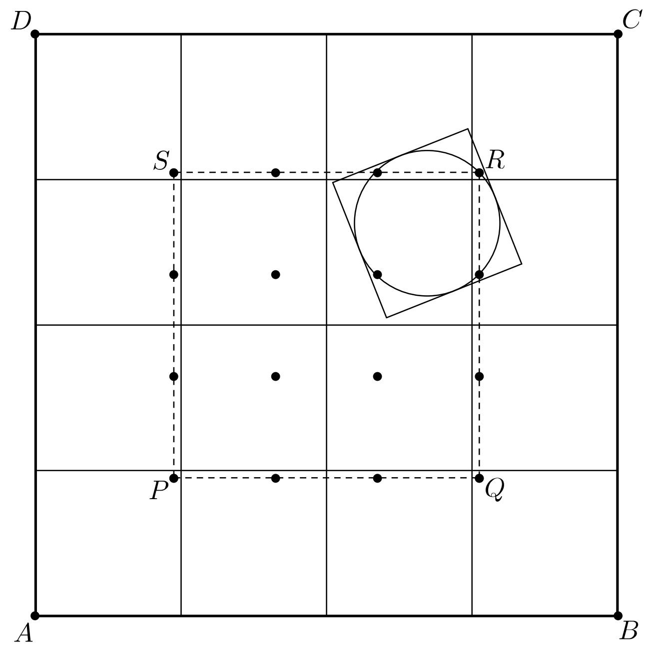

Figure 1: 16 points in 4×4 square.

that, for each point inside the square $PQRS$, there exists at least one of the chosen 16 points at a distance from the observed point of no more than $\frac{\sqrt{2}}{2} \cdot \frac{2+2\varepsilon}{3} = \frac{\sqrt{2}}{3}(1+\varepsilon)$. Therefore, if ε is small enough, each circle centered inside the square $PQRS$ with radius $\frac{1}{2}$ contains at least one of the chosen 16 points, and thus the same clearly holds for any unit square whose center is inside the square $PQRS$.

Let us now consider unit squares whose center is not inside the square $PQRS$. Suppose that we have such a square that does not contain any of the chosen 16 points in its interior. We can, w.l.o.g., assume that one of the following two cases holds:

- two neighboring edges of the considered unit square contain one of the chosen 16 points each, and in fact ones from the edges of the square $PQRS$ (Figure 2 left);
- two neighboring vertices of the considered unit square belong to two neighboring edges of the square $ABCD$ (Figure 2 right).

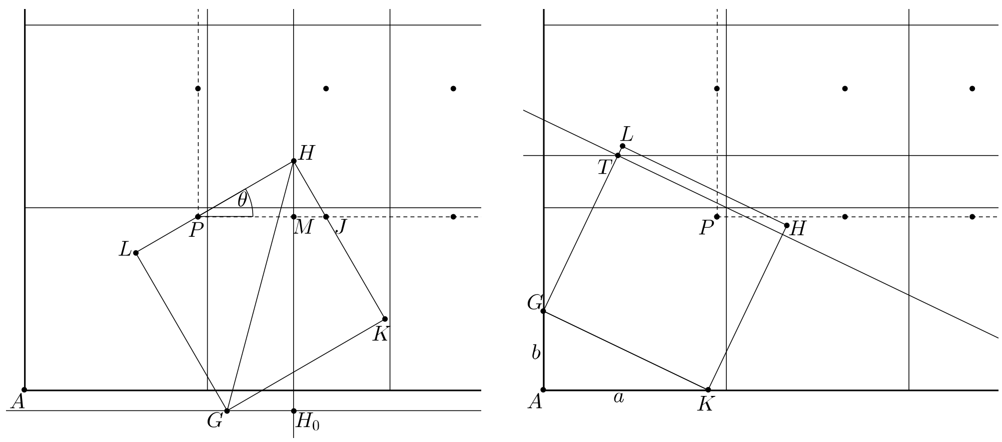

Figure 2: The two possible cases.

Let us first deal with the former case. Let $KHLG$ be a unit square and let its sides HL and HK contain the points P and J (these are two of the chosen 16 points, from the edges of the square $PQRS$), respectively. Denote $\angle HPJ = \theta$ (we can assume $\theta \leqslant 45^\circ$). Let M be the foot of the perpendicular from H to PJ, and let H_0 be the foot of the perpendicular from G to the line HM. We shall prove that the vertex G is outside the square $ABCD$; it is enough to show $\overline{HH_0} > (1-\varepsilon)+\overline{MH}$. Since also $\angle JHM = \theta$, we evaluate

$$\overline{MH} = \overline{JH}\cos\theta = \overline{JP}\sin\theta\cos\theta = \frac{2+2\varepsilon}{3}\sin\theta\cos\theta.$$

Further, we have $\angle GHH_0 = 45^\circ - \angle JHM = 45^\circ - \theta$, from which follows

$$\begin{aligned}\overline{HH_0} &= \overline{HG}\cos\angle GHH_0 = \sqrt{2}\cdot\cos(45^\circ-\theta)\\ &= \sqrt{2}(\cos 45^\circ\cos\theta + \sin 45^\circ \sin\theta) = \cos\theta + \sin\theta.\end{aligned}$$

Therefore, we actually need to prove the inequality

$$\cos\theta + \sin\theta - (1-\varepsilon) - \frac{2+2\varepsilon}{3}\sin\theta\cos\theta > 0. \tag{1}$$

Notice that:

$$\begin{aligned}
&\cos\theta+\sin\theta-(1-\varepsilon)-\frac{2+2\varepsilon}{3}\sin\theta\cos\theta\\
&\quad=\cos\theta+\sin\theta-(1-\varepsilon)-\frac{1+\varepsilon}{3}(\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta)+\frac{1+\varepsilon}{3}\\
&\quad=\cos\theta+\sin\theta-\frac{2-4\varepsilon}{3}-\frac{1+\varepsilon}{3}(\sin\theta+\cos\theta)^2.
\end{aligned}$$

Zeros of the quadratic function

$$f(x)=-\frac{1+\varepsilon}{3}x^2+x-\frac{2-4\varepsilon}{3}$$

are

$$\begin{aligned}
x_{1,2}&=\frac{-1\pm\sqrt{1-4\cdot\frac{1+\varepsilon}{3}\cdot\frac{2-4\varepsilon}{3}}}{-2\cdot\frac{1+\varepsilon}{3}}=\frac{-3\pm\sqrt{9-4(2-2\varepsilon-4\varepsilon^2)}}{-2(1+\varepsilon)}\\
&=\frac{-3\pm\sqrt{16\varepsilon^2+8\varepsilon+1}}{-2(1+\varepsilon)}=\frac{3\mp(4\varepsilon+1)}{2(1+\varepsilon)},
\end{aligned}$$

that is, the function f is positive for $x\in\left(\frac{1-2\varepsilon}{1+\varepsilon},2\right)$. Since for $0\leqslant\theta\leqslant 45^\circ$ we have $\sin\theta+\cos\theta<2$ and $\sin\theta+\cos\theta\geqslant\sin^2\theta+\cos^2\theta=1>\frac{1-2\varepsilon}{1+\varepsilon}$, this completes the proof in the first case.

Let us now consider the second case. Let $\overline{AK}=a$, $\overline{AG}=b$ (where $a^2+b^2=1$). The equation of the line GL is $y=\frac{a}{b}x+b$. Consider the perpendicular from the point $(1,1)$ to the line GL. The equation of this perpendicular is $y=-\frac{b}{a}x+1+\frac{b}{a}$. Solving the system of the last two equations gives $x=\frac{1+\frac{b}{a}-b}{\frac{a}{b}+\frac{b}{a}}=\frac{(a+b-ab)b}{a^2+b^2}=ab+b^2-ab^2$ and $y=\frac{a}{b}(ab+b^2-ab^2)+b=a^2+ab-a^2b+b$, that is, the considered perpendicular intersects the line GL at the point T with the coordinates $(ab+b^2-ab^2,a^2+ab-a^2b+b)$. We claim that $\overline{GT}<1$. Indeed,

$$\begin{aligned}
\overline{GT}^2&=(a^2+ab-a^2b)^2+(ab+b^2-ab^2)^2\\
&=a^2(a+b-ab)^2+b^2(a+b-ab)^2=(a+b-ab)^2(a^2+b^2)\\
&=(a+b-ab)^2;
\end{aligned}$$

therefore, we need to prove $a + b - ab < 1$, that is, $(1 - a)(1 - b) > 0$, which is clearly true.

From $\overline{GT} < 1$ we get that the point $(1, 1)$ lies in the interior of the square $KHLG$, and therefore so does the point P, a contradiction.

This completes the proof for $n = 4$. For $n < 4$ it is enough to take any $n \times n$ square inside the square in Figure 1 containing n^2 of the chosen 16 points in its interior. ■

3 An upper bound

Let us now refine the methods from the previous section in order to obtain an upper bound that we believe is quite strong. We first prove a lemma that will be useful.

Lemma 2. *Let the points U and V be chosen on two neighboring edges, say GK and GL, of a unit square $KHLG$, such that $\overline{UV} < 1$. Let W be the third vertex of the equilateral triangle UVW, where the point W is on the same side of the line UV as the point H. Then the point W lies in the interior of the square $KHLG$.*

Proof. It is enough to show that the point W is in the same open halfplanes as the points V and U with respect to the lines KH and LH, respectively (Figure 3). And indeed, this directly follows from the observations $\overline{VW} < 1$ and $\overline{UW} < 1$, respectively. ■

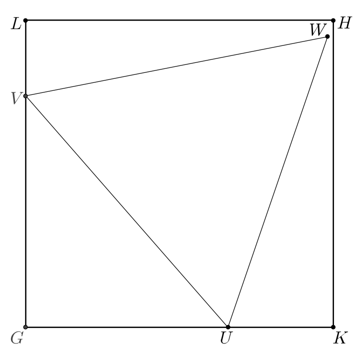

Figure 3: Picture for Lemma 2.

Let us now show the main result of this section.

Theorem 3. *For any $n \in \mathbb{N}$,*

$$\pi(\mathscr{U}_n) \leqslant n \left(\left\lceil \frac{2}{\sqrt{3}}(n + 1 - 2\sqrt{2}) \right\rceil + 1 \right). \tag{2}$$

Proof. Let the vertices A, B, C, D of the square $ABCD$ have the coordinates $(0,0)$, $(n,0)$, (n,n) and $(0,n)$, respectively. We shall first place an equilateral triangular lattice in the square $ABCD$, as sketched in Figure 4.

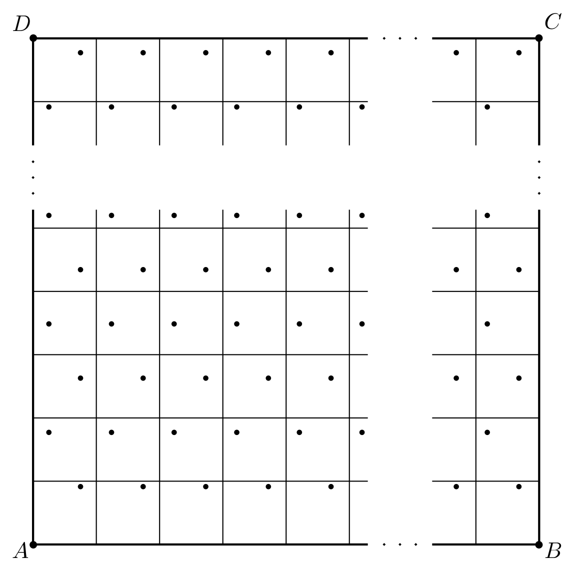

Figure 4: Equilateral triangular lattice.

The lattice is defined as follows:

i) the side length of the equilateral triangle that generates this lattice is $1 - \delta$, where δ is a small enough positive number;

ii) the bottom left point is at the coordinates $(\frac{3}{4}, \sqrt{2} - \frac{1}{2})$ (the value $\frac{3}{4}$ here is not really important, the proof would be completely the same for any number from the interval $(\frac{1}{2}, 1)$; on the other hand, the motivation for the value $\sqrt{2} - \frac{1}{2}$ comes from the fact that this is the largest number such that the inequality (4) further in the proof holds for all $\delta > 0$ and $0 \leqslant \theta \leqslant 45°$);

iii) all the points in the bottom "row" are on a line parallel to the x-axis (and the same holds for further rows);

iv) the number of rows is determined so that the y-coordinate of the points in the top row is greater than or equal to $n - (\sqrt{2} - \frac{1}{2})$.

To state it in a more formal (but possibly less clear) way: we arrange points in a total of $\left\lceil \frac{n-2(\sqrt{2}-\frac{1}{2})}{\frac{\sqrt{3}}{2}(1-\delta)} \right\rceil + 1$ rows (the number of rows follows from the requirements ii) and iv) and the fact that the distance between two successive rows equals $\frac{\sqrt{3}}{2}(1-\delta)$); in the j^{th} row from the bottom there are n points having the coordinates

$$\Big(\frac{3}{4} + i(1-\delta), \sqrt{2} - \frac{1}{2} + (j-1)\frac{\sqrt{3}}{2}(1-\delta)\Big),\ 0 \leqslant i \leqslant n-1$$

if j is odd, and

$$\Big(\frac{3}{4} - \frac{1-\delta}{2} + i(1-\delta), \sqrt{2} - \frac{1}{2} + (j-1)\frac{\sqrt{3}}{2}(1-\delta)\Big),\ 0 \leqslant i \leqslant n-1$$

if j is even.

Since $\lim\limits_{\delta \to 0} \frac{n-2(\sqrt{2}-\frac{1}{2})}{\frac{\sqrt{3}}{2}(1-\delta)} = \frac{2}{\sqrt{3}}(n+1-2\sqrt{2})$, which is clearly not an integer, we have

$$\lim_{\delta \to 0} \left(\left\lceil \frac{n-2(\sqrt{2}-\frac{1}{2})}{\frac{\sqrt{3}}{2}(1-\delta)} \right\rceil + 1 \right) = \left\lceil \frac{2}{\sqrt{3}}(n+1-2\sqrt{2}) \right\rceil + 1. \tag{3}$$

Therefore, for δ small enough, the total number of chosen points equals $n\big(\lceil \frac{2}{\sqrt{3}}(n+1-2\sqrt{2}) \rceil + 1\big)$ (note that this is precisely the right-hand side of (2)). Also note that, for δ small enough, if further points were added to any of the rows (following the same pattern), they would be outside the square $ABCD$. Let us show that there indeed does not exist a unit square inside the square $ABCD$ containing none of the chosen points in its interior.

Suppose the opposite: let $KHLG$ be a unit square inside the square $ABCD$ such that none of the chosen points are in its interior. We can, w.l.o.g., assume that one of the following three cases holds:

- two neighboring edges of the square $KHLG$ contain one of the chosen points each, where both of these points simultaneously belong to the bottom or to the top row;
- two neighboring edges of the square $KHLG$ contain one of the chosen points each, where not both of these points simultaneously belong to the bottom or to the top row;
- two neighboring vertices of the square $KHLG$, say K and G, belong to two neighboring edges of the square $ABCD$, and furthermore, the one of the chosen points that is in the relevant corner lies between the lines KH and GL.

The third case may need some explanation. By "the one of the chosen points that is in the relevant corner" we mean the following: the "relevant corner" is the one where the two edges of the square $ABCD$ on which the points K and G lie meet (e.g., if $K \in AB$ and $G \in AD$, then the "relevant corner" is bottom-left); the point that is required to be between the lines KH and GL is the leftmost or the rightmost point in the bottom or the top row, whichever of these is determined by the "relevant corner" (e.g., if the "relevant corner" is top-left, then the leftmost point in the top row should be between the lines KH and GL). Note that we can indeed assume this additional constraint in the third case, since if it does not hold, then the square $KHLG$ can be moved to fall under one of the first two cases.

Having the definition of the third case cleared, we note that it leads to a contradiction in the same way as in the corresponding part in Theorem 1 (the one that is shown in Figure 2 right), verbatim. Assume now the second case. Let U and V be two of the chosen points such that $U \in GK$, $V \in GL$. By Lemma 2, the third vertex W of the equilateral triangle UVW lies in the interior of the square $KHLG$. However, we note that the point W is either also one of the chosen points, or lies outside the square $ABCD$ (the latter possibility could happen if U and V are the leftmost or the rightmost points of two consecutive rows; we here recall that δ could be arbitrarily small). One way or another, we reach a contradiction. That leaves only the first case.

Note that, in the first case, if the center of the square $KHLG$ is in the strip between the bottom and the top row, we reach a contradiction in the same way as in the second case. Therefore, assume that the center of the square $KHLG$ lies below the bottom or above the top row. We shall follow the lines of thought from Theorem 1 (we here have the picture that is like the one shown in Figure 2 left). We evaluate

$$\begin{aligned}\overline{MH} &= \overline{JP}\sin\theta\cos\theta = (1-\delta)\sin\theta\cos\theta;\\ \overline{HH_0} &= \cos\theta + \sin\theta.\end{aligned}$$

Therefore, the inequality corresponding to (1) (that we need to prove) is

$$\cos\theta + \sin\theta - \left(\sqrt{2} - \frac{1}{2}\right) - (1-\delta)\sin\theta\cos\theta > 0. \tag{4}$$

Notice that:

$$\begin{aligned}\cos\theta + \sin\theta &- \left(\sqrt{2} - \frac{1}{2}\right) - (1-\delta)\sin\theta\cos\theta\\ &= \cos\theta + \sin\theta - \left(\sqrt{2} - \frac{1}{2}\right)\\ &\qquad - \frac{1-\delta}{2}(\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta) + \frac{1-\delta}{2}\\ &= \cos\theta + \sin\theta - \left(\sqrt{2} - 1 + \frac{\delta}{2}\right) - \frac{1-\delta}{2}(\sin\theta + \cos\theta)^2,\end{aligned}$$

We know $\cos\theta + \sin\theta \geqslant 1$ (because θ is nonnegative). Let us check what is the maximal possible value of $\cos\theta + \sin\theta$. Since $(\cos\theta + \sin\theta)' = -\sin\theta + \cos\theta$, equating this with 0 shows that the extrema are reached for $\theta = \pm\frac{\pi}{4} = \pm 45°$; in particular, the local maximum (which is also the global maximum) is reached for $\theta = 45°$ (which indeed satisfies our constraint $\theta \leqslant 45°$) and equals $\sqrt{2}$. Therefore, since $1 \leqslant \cos\theta + \sin\theta \leqslant \sqrt{2}$, and since the quadratic function

$$g(x) = -\frac{1-\delta}{2}x^2 + x - \left(\sqrt{2} - 1 + \frac{\delta}{2}\right)$$

has negative leading coefficient, in order to finish the proof it is enough to check that $g(1) > 0$ and $g(\sqrt{2}) > 0$. And indeed:

$$\begin{aligned}g(1) &= -\frac{1-\delta}{2} + 1 - \sqrt{2} + 1 - \frac{\delta}{2} = \frac{3}{2} - \sqrt{2} > 0;\\ g(\sqrt{2}) &= -\frac{1-\delta}{2}\cdot 2 + \sqrt{2} - \sqrt{2} + 1 - \frac{\delta}{2} = \frac{\delta}{2} > 0.\end{aligned}$$

This completes the proof. ■

n	1	2	3	4	5	6	7	8	9	10
upper bound	1	4	9	16	25	36	49	72	90	110
n	11	12	13	14	15	16	17	18	19	20
upper bound	132	156	182	224	255	288	323	360	399	440

Table 1: Upper bounds for $\pi(\mathscr{U}_n)$ for $n \leqslant 20$.

The exact upper bounds from Theorem 3 for $n \leqslant 20$ are calculated in Table 1. These values give the following corollary.

Corollary 4. *For $n \leqslant 7$,*

$$\pi(\mathscr{U}_n) = n^2.$$

4 Some versions of the problem

We here mention some versions of the problem for which the results follow by slight alterations of the reasoning presented so far.

First, we can ask the same questions for rectangles $ABCD$ instead of squares. Let $\mathscr{U}_{m,n}$ denote the collection of all open unit squares than can be fitted inside a given $m \times n$ rectangle. Then $\pi(\mathscr{U}_{m,n})$ stands for the minimum number of points that need to be chosen inside an $m \times n$ rectangle $ABCD$ such that there does not exist a unit square inside the rectangle $ABCD$ containing none of the chosen points in its interior. A straightforward modification of the proof of Theorem 3 gives the following result.

Theorem 5. *For each $m, n \in \mathbb{N}$,*

$$\pi(\mathscr{U}_{m,n}) \leqslant \min\left\{m\left(\left\lceil \frac{2}{\sqrt{3}}(n+1-2\sqrt{2})\right\rceil + 1\right), n\left(\left\lceil \frac{2}{\sqrt{3}}(m+1-2\sqrt{2})\right\rceil + 1\right)\right\}.$$

Of course, the lower bound for $\pi(\mathscr{U}_{m,n})$ is mn. Since $\lceil \frac{2}{\sqrt{3}}(n+1-2\sqrt{2})\rceil + 1 = n$ for each $n \leqslant 7$, we have an interesting corollary that this lower bound is matched for all the rectangles whose one side is of length at most 7, no matter how long the other side is! In other words:

Corollary 6. *If* $\min\{m,n\} \leqslant 7$*, then*

$$\pi(\mathscr{U}_{m,n}) = mn.$$

Let us now find an upper bound for $\pi(\mathscr{U}_x)$ when x is not necessarily an integer.

Theorem 7. *For any* $x > 0$*,*

$$\pi(\mathscr{U}_x) \leqslant \begin{cases} \lfloor x \rfloor \left(\left\lfloor \frac{2}{\sqrt{3}}(x+1-2\sqrt{2}) \right\rfloor + 2 \right), & \text{if } \{x\} < \frac{1}{2}; \\[2ex] \lfloor x \rfloor \left(\left\lfloor \frac{2}{\sqrt{3}}(x+1-2\sqrt{2}) \right\rfloor + 2 \right) + \left\lfloor \frac{\left\lfloor \frac{2}{\sqrt{3}}(x+1-2\sqrt{2}) \right\rfloor + 2}{2} \right\rfloor, & \\ \hfill \text{if } \{x\} \geqslant \frac{1}{2}. \end{cases}$$

(Hereby $\{x\}$ *denotes the fractional part of* x*, that is,* $\{x\} = x - \lfloor x \rfloor$*.)*

Proof. The proof is basically the same as the proof of Theorem 3. We highlight only the necessary modifications.

We construct an equilateral triangular lattice with the step $1-\delta$ as in the proof of Theorem 3, but with the bottom left point at the coordinates $(c, \sqrt{2} - \frac{1}{2})$, where c is chosen so that $c > \frac{1}{2}$ and $\{x\} < c < 1$. Therefore, for δ small enough, the first row, as well as all the odd rows, consist of $\lfloor x \rfloor$ points. The leftmost point in the second row has the x-coordinate equal to $c - \frac{1-\delta}{2}$, which is, for δ small enough, smaller than $\frac{1}{2}$ but can be arbitrarily close to $\frac{1}{2}$ (since c can be chosen arbitrarily close to 1). Therefore, if δ is small enough and c is close to 1, the x-coordinates of the first $\lfloor x \rfloor$ points from the second row are in the intervals $(i, i+\frac{1}{2})$, where i ranges from 0 to $\lfloor x \rfloor - 1$, and actually can be made arbitrarily close to $i + \frac{1}{2}$. These are actually all points from the second row if and only if $x - (\lfloor x \rfloor - 1 + \frac{1}{2}) < 1$, that is, if and only if $\{x\} < \frac{1}{2}$; otherwise, the second row has one more point. The same holds for the fourth, the sixth etc., that is, for all even rows. Altogether, if d denotes the number of rows, the total number of chosen points equals $\lfloor x \rfloor d$ if $\{x\} < \frac{1}{2}$ (d rows with $\lfloor x \rfloor$ points each), and $\lfloor x \rfloor d + \lfloor \frac{d}{2} \rfloor$ if $\{x\} \geqslant \frac{1}{2}$ (d rows with $\lfloor x \rfloor$ points each, plus one additional point in each of $\lfloor \frac{d}{2} \rfloor$ rows).

That leaves only to calculate d, the number of rows. As in (3), we have that, for δ small enough, the number of rows equals $\lceil \frac{2}{\sqrt{3}}(x+1-2\sqrt{2}) \rceil + 1$ if $\frac{2}{\sqrt{3}}(x+1-2\sqrt{2})$ is not an integer, but equals $\lceil \frac{2}{\sqrt{3}}(x+1-2\sqrt{2}) \rceil + 2$ if $\frac{2}{\sqrt{3}}(x+1-2\sqrt{2})$ is an integer. These two formulas actually can be unified by concluding that the number of rows equals $\lfloor \frac{2}{\sqrt{3}}(x+1-2\sqrt{2}) \rfloor + 2$. Together with the conclusion at the end of the previous paragraph, we are now able to calculate the total number points, which gives exactly the upper bound from the statement of the theorem. ■

5 An application to the square packing problem

For a positive integer n, let $s(n)$ denote the side length of the smallest square into which n unit squares can be packed. Trivial bounds for $s(n)$ are $\sqrt{n} \leqslant s(n) \leqslant \lceil \sqrt{n} \rceil$. The cases in which the exact values of $s(n)$ are known, and some bounds for other cases, are summarized in a dynamic survey by Friedman [11].

The following proposition makes a connection between our problem and the square packing problem.

Proposition 8. *For any $x > 0$, no more than $\pi(\mathscr{U}_x)$ unit squares can be packed in a square of side length x.*

Proof. Let a square of side length x be given. Choose $\pi(\mathscr{U}_x)$ points in its interior such that there does not exist a unit square inside the given square containing none of the chosen points in its interior. Therefore, if more than $\pi(\mathscr{U}_x)$ unit squares were packed inside the given square, there would exist two of them containing in their interiors the same point among the chosen $\pi(\mathscr{U}_x)$ points, that is, their interiors would have a nonempty intersection, which is a contradiction. ■

In other words, the proposition states

$$s(\pi(\mathscr{U}_x) + 1) > x \tag{5}$$

for any $x > 0$. This inequality can be used to reprove the following results.

Theorem 9. *The values of $s(n)$ for some values of n are as given in Table 2.*

n	$s(n)$	originally proved by	n	$s(n)$	originally proved by
8	3	Bajmóczy, by [13]	35	6	Friedman [11]
15	4	El Moumni [9]	46	7	Bentz [2]
23	5	Nagamochi [20]	47	7	Nagamochi [20]
24	5	Friedman [11]	48	7	Nagamochi [20]
34	6	Nagamochi [20]			

Table 2: Values of $s(n)$ proved in Theorem 9.

Proof. We show only $s(46) = 7$. The other proofs are completely analogous.

For each small enough $\varepsilon > 0$ we evaluate

$$\lfloor 7 - \varepsilon \rfloor = 6$$

and

$$\left\lfloor \frac{2}{\sqrt{3}}((7-\varepsilon) + 1 - 2\sqrt{2}) \right\rfloor = 5.$$

Therefore, Theorem 7 now enables us to evaluate

$$\pi(\mathscr{U}_{7-\varepsilon}) \leqslant 6 \cdot (5+2) + \left\lfloor \frac{5+2}{2} \right\rfloor = 42 + 3 = 45.$$

The function s is nondecreasing, and thus (5) gives

$$s(46) = s(45+1) \geqslant s(\pi(\mathscr{U}_{7-\varepsilon}) + 1) > 7 - \varepsilon.$$

Since the above inequality holds for each small enough $\varepsilon > 0$, we deduce

$$s(46) \geqslant 7.$$

On the other hand, 46 unit squares can be easily packed in a square of side length 7. This proves $s(46) = 7$. ∎

Finally, for those values of n for which no exact value of $s(n)$ is known, obtaining some (nontrivial) upper and lower bounds on it is an interesting research direction. The list of the best known such bounds is compiled in the already mentioned survey [11]. By the approach from the present paper, we were able to improve the lower bound for $s(61)$ (until now no nontrivial lower bound on $s(61)$, that is, better than $s(61) \geqslant \sqrt{61} \approx 7.8102$, was known).

Theorem 10. *We have*

$$s(61) \geqslant \frac{7\sqrt{3}}{2} + 2\sqrt{2} - 1 \approx 7.8906.$$

Proof. The idea is the same as in the proof of the previous theorem. For each small enough $\varepsilon > 0$ we evaluate

$$\left\lfloor \frac{7\sqrt{3}}{2} + 2\sqrt{2} - 1 - \varepsilon \right\rfloor = 7$$

and

$$\left\lfloor \frac{2}{\sqrt{3}} \left(\left(\frac{7\sqrt{3}}{2} + 2\sqrt{2} - 1 - \varepsilon \right) + 1 - 2\sqrt{2} \right) \right\rfloor = \left\lfloor \frac{2}{\sqrt{3}} \left(\frac{7\sqrt{3}}{2} - \varepsilon \right) \right\rfloor = 6,$$

which leads to

$$\pi(\mathscr{U}_{\frac{7\sqrt{3}}{2}+2\sqrt{2}-1-\varepsilon}) \leqslant 7 \cdot (6+2) + \left\lfloor \frac{6+2}{2} \right\rfloor = 56 + 4 = 60.$$

This implies $s(61) \geqslant \frac{7\sqrt{3}}{2} + 2\sqrt{2} - 1$ in the same way as in the previous theorem. ■

6 A conjecture about asymptotical tightness

For the end, let us say that, though exact formula for $\pi(\mathscr{U}_n)$ (or $\pi(\mathscr{U}_x)$) may not be easy to find, we believe that the upper bound (2) is asymptotically tight.

Conjecture. *For $n \to \infty$, we have*

$$\pi(\mathscr{U}_n) \sim \frac{2}{\sqrt{3}} n^2.$$

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References

[1] N. Alon & D. J. Kleitman, Piercing convex sets and the Hadwiger-Debrunner (p, q)-problem, *Adv. Math.* **96** (1992), 103–112.

[2] W. Bentz, Optimal packings of 13 and 46 unit squares in a square, *Electron. J. Combin.* **17** (2010), Research Paper #R126, 11 pp.

[3] H. T. Croft & K. J. Falconer & R. K. Guy, *Unsolved Problems in Geometry*, Springer-Verlag, New York, 1994.

[4] L. Danzer, Ungelöste Probleme, Nachtrag zu Nr. 15, *Elem. Math.* **12** (1957), 62.

[5] L. Danzer, Zur Lösung des Gallaischen Problems über Kreisscheiben in der euklidischen Ebene, *Studia Sci. Math. Hungar.* **21** (1986), 111–134.

[6] L. Danzer & B. Grünbaum & V. Klee, Helly's theorem and its relatives, in: V. Klee (ed.), *Convexity*, American Mathematical Society, Providence, Rhode Island, 1963, pp. 101–180.

[7] J. Eckhoff, A survey of the Hadwiger-Debrunner (p, q)-problem, in: B. Aronov & S. Basu & J. Pach & M. Sharir (eds.), *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, Springer-Verlag, Berlin, 2003, pp. 347–377.

[8] J. Eckhoff, Helly, Radon, and Carathéodory type theorems, in: P. M. Gruber & J. M. Wills (eds.), *Handbook of Convex Geometry, Vol. A*, North-Holland, Amsterdam, 1993, pp. 389–448.

[9] S. El Moumni, Optimal packings of unit squares in a square, *Studia Sci. Math. Hungar.* **35** (1999), 281–290.

[10] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer-Verlag, Berlin, 1953.

[11] E. Friedman, Packing unit squares in squares: a survey and new results, *Electron. J. Combin.*, Dynamic Survey #DS7.

[12] E. Friedman & D. A. Paterson, Covering squares with unit squares, *Geombinatorics* **15** (2006), 130–137.

[13] F. Göbel, Geometrical packing and covering problems, in: A. Schrijver (ed.), *Packing and Covering in Combinatorics*, Mathematisch Centrum, Amsterdam, 1979, pp. 179–199.

[14] B. Grünbaum, On intersections of similar sets, *Portugal. Math.* **18** (1959), 155–164.

[15] H. Hadwiger & H. Debrunner, Über eine Variante zum Hellyschen Satz, *Arch. Math. (Basel)* **8** (1957), 309–313.

[16] R. N. Karasev, Transversals for families of translates of a two-dimensional convex compact set, *Discrete Comput. Geom.* **24** (2000), 345–353.

[17] D. J. Kleitman & A. Gyárfás & G. Tóth, Convex sets in the plane with three of every four meeting, *Combinatorica* **21** (2001), 221–232.

[18] L. Moser, Problem 24, *Canad. Math. Bull.* **3** (1960), 78.

[19] W. O. J. Moser, Problems on extremal properties of a finite set of points, in: J. E. Goodman, & E. Lutwak & J. Malkevitch & R. Pollack (eds.), *Discrete Geometry and Convexity*, *Ann. New York Acad. Sci.* **440** (1985), pp. 52–64.

[20] H. Nagamochi, Packing Unit Squares in a Rectangle, *Elect. J. Comb.* **12** (2005), Research Paper #R37, 13 pp.

[21] J. Pach & G. Tardos, Piercing quasi-rectangles—on a problem of Danzer and Rogers, *J. Combin. Theory Ser. A* **119** (2012), 1391–1397.

[22] L. Stachó, A Gallai-féle körletüzési probléma megoldása, *Mat. Lapok* **32** (1981/84), 19–47.

[23] P. G. Szabó & M. Cs. Markót & T. Csendes & E. Specht & L. G. Casado & I. García, *New Approaches to Circle Packing in a Square*, Springer, New York, 2007.